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Open/closed string duality for topological gravity with matter

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Abstract: The exact FZZT brane partition function for topological gravity with matter is computed using the dual two-matrix model. We show how the effective theory of open strings on a stack of FZZT branes is described by the generalized Kontsevich matrix integral, extending the earlier result for pure topological gravity. Using the well-known relation between the Kontsevich integral and a certain shift in the closed-string background, we conclude that these models exhibit open/closed string duality explicitly. Just as in pure topological gravity, the unphysical sheets of the classical FZZT moduli space are eliminated in the exact answer. Instead, they contribute small, nonperturbative corrections to the exact answer through Stokes' phenomenon.

Keywords: Integrable Hierarchies, Matrix Models.

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1. Introduction

Open/closed string duality is one of the most intriguing facets of string theory. It states that dynamical processes involving open strings can be formulated strictly in terms of closed strings, and vice versa. Often, the open string side of the correspondence can be reduced to an ordinary gauge theory. In such cases, the correspondence can provide a promising way to address outstanding issues of quantum gravity by reformulating them as questions in gauge theory.

Recently, there has been renewed interest in "minimal string theory", i.e. non-critical strings with $c < 1$. (For a list of references to recent work, see e.g. [1].) These models are dual to certain zero-dimensional gauge theories (i.e. matrix models), providing a tantalizing hint that some form of open/closed string duality is at work here. Because these systems are integrable, it is natural to expect that $c < 1$ string theories exhibit open/closed string dualities which are explicitly demonstrable. This strongly motivates us to study these models further.

One of the main ingredients in formulating an open/closed string duality are D-branes. The D-branes relevant to non-critical strings with $c < 1$ are called FZZT branes, and they were studied in the context of Liouville theory in [2, 3]. In the recent papers of [4 – 6], FZZT branes were used to provide a geometric interpretation of minimal string theory, as well as to probe the structure of its target space. Meanwhile, they were used in [7] to explore the relation of minimal strings to topological strings on non-compact Calabi-Yaus with B-branes.

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Since our goal here is to understand open/closed string duality in minimal string theory, it is natural to consider the dynamics governing the open strings ending on FZZT branes. Exactly this question was addressed for the case of the $c = -2$ model — known as pure topological gravity from the work of [8] — first from the point of view of string field theory in [9], and later using the double-scaled matrix model in [6]. Both approaches found that the open strings ending on FZZT branes in this background are described by the matrix integral of Kontsevich [10]. By identifying the precise deformation of the closed string background associated with the presence of the brane, the authors of [6, 9] showed concretely how Kontsevich's formulation of two dimensional gravity can be thought of as a kind of open/closed string duality.

Pure topological gravity is often referred to as the $(2, 1)$ model, by analogy with (p, q) minimal string theory. It has natural generalizations, commonly referred to as the $(p, 1)$ models, which describe topological matter coupled to gravity. The appropriate generalizations of the Kontsevich matrix integral also exist [10, 11]. In this article, we will use the double-scaled two-matrix model [12] to relate the two descriptions via open/closed string duality. That is, we will show explicitly that the generalized Kontsevich integral describes the effective theory of open strings between FZZT branes in the $(p, 1)$ closed-string background. Just as for the $(2, 1)$ model, we will see that the generalized Kontsevich integral includes nonperturbative effects that drastically modify the topology of the FZZT moduli space.

This paper is organized as follows. We begin in section 2 by formulating the $(p, 1)$ minimal string theories as a double-scaling limit of the two matrix model. We compute the partition function of a stack of FZZT branes in this background and obtain the generalized (matrix) Airy function. We also explore the Stokes and anti-Stokes lines for the FZZT partition function. In section 3, we use the correspondence between macroscopic loop operators and the local closed string operators to formulate the open/closed string duality for the $(p, 1)$ model. Finally, we conclude in section 4 with a discussion of open problems and relation to other work. The appendices contain various generalizations and technical details.

2. Double-scaling the $(p, 1)$ models

2.1 FZZT branes in the two-matrix model

In this section, we describe the analysis of the FZZT partition function for general $(p, 1)$ models using the two-matrix model, extending the results of [6]. Let us begin by considering a general form of the two-matrix model

$$\mathcal{Z}(g) = \int dA\, dB\, e^{-\frac{1}{g}\mathrm{Tr}(V(A)+W(B)-AB)}\,, \tag{2.1}$$

where A and B are $N \times N$ matrices, g is the coupling constant of the bulk theory, and the choice of integration contour depends on the form of the potentials V and W. This model can describe (p, q) minimal string theory in the large-N limit [12], provided we tune the potentials $V(A)$ and $W(B)$ to

$$\begin{aligned} V'(a) &= \frac{1}{2\pi i}\oint \frac{Y_*(1/z)}{a - X_*(z)} dX_*(z) \\ W'(b) &= \frac{1}{2\pi i}\oint \frac{X_*(1/z)}{b - Y_*(z)} dY_*(z)\,, \end{aligned} \tag{2.2}$$

where we integrate z on a small contour around $z = 0$, and $X_*(z)$ and $Y_*(z)$ have zeros of order q and p, respectively, at $z = 1$. We find it convenient to take

$$\begin{aligned} X_*(z) &= \frac{(z-1)^q}{z} \\ Y_*(z) &= \frac{(1-z)^p}{z}\,. \end{aligned} \tag{2.3}$$

Then, for $q = 1$, the potentials $V(a)$ and $W(b)$ take the simple form

$$V_p(a) = -\sum_{k=1}^{p} \frac{a^k}{k} + H_p\,, \qquad W_p(b) = b\,. \tag{2.4}$$

Here we have fixed the integration constant to be

$$H_p = \sum_{k=1}^{p} \frac{1}{k} \tag{2.5}$$

so that $V_p(1) = 0$. This will be convenient for later calculations. One can further simplify the potential by shifting $A \to A + 1$, which leads to

$$\mathcal{Z}_{(p,1)}(g) = \int dA\, dB\, e^{-\frac{1}{g}\operatorname{Tr}(V_p(A+1) - AB)}\,. \tag{2.6}$$

In the large-N double-scaling limit, this becomes the bulk partition function of the $(p, 1)$ model. Two comments on this result are in order:

1. For $p = 2$, one can integrate out A to obtain a gaussian model for B. This case was studied in detail in [6]. Of course, A cannot be integrated out so easily in general. Nevertheless, the integral (2.6) is essentially trivial for any value of p, since B always acts like a Lagrange multiplier constraining A. This will facilitate much of the computation reported in this article.

2. Some care is necessary in order to ensure that the matrix integral in (2.6) is well defined. For even p, this can be accomplished by integrating A and B with respect to the measure where ηA and $i\eta^{-1} B$ are hermitean matrices, and $\eta^p = -1$. For odd p, one can use the same measure, but one should integrate first over B so that the A integral is constrained.

We will consider exact D-brane observables constructed from the macroscopic loop operator [13–15]

$$W(y) = \operatorname{Tr}\log(y - B)\,. \tag{2.7}$$

Here y is called the boundary cosmological constant, and it parametrizes the moduli space of the FZZT brane. Following [6], the simplest such observable is the exact partition function of the FZZT brane:

$$\left\langle e^{W(y)} \right\rangle = \langle \det(y-B) \rangle = \mathcal{Z}_{p,1}(g)^{-1} \int dA\, dB\, e^{-\frac{1}{g} \operatorname{Tr}(V_p(A+1)-AB)} \det(y-B)\,. \tag{2.8}$$

Here the integral over A and B is defined in the same way as in (2.6), as discussed above. For the rest of this section, we will focus mainly on the single determinant expectation value (2.8), since this captures the essentials of the matrix model analysis. In section 2.4 we will also extend our analysis to the correlator of multiple determinants.

The expectation value of $\det(y-B)$ can be computed efficiently using the method of orthogonal polynomials (for a review and original references, see e.g. [16, 17]). The bi-orthogonal polynomials are defined with the normalization $P_m(a) = a^m + \ldots$ and $Q_n(b) = b^n + \ldots$, and satisfy the orthogonality relation

$$\int da\, db\, e^{-\frac{1}{g}(V_p(a+1)-ab)} P_m(a) Q_n(b) = h_m \delta_{m,n}\,, \tag{2.9}$$

where again, the integration over a and b is defined as in (2.6). For the orthogonality relation (2.9), it is possible to write down explicit, closed-form expressions for P_n and Q_n:

$$\begin{aligned} P_n(a) &= a^n \\ Q_n(b) &= \left(-g\frac{\partial}{\partial z}\right)^n e^{\frac{1}{g}(V_p(z+1)-bz)}\bigg|_{z=0}\,. \end{aligned} \tag{2.10}$$

It is straightforward to verify that (2.10) satisfies (2.9).

In terms of these orthogonal polynomials, the expectation value of the determinant is simply

$$\langle \det(y-B) \rangle = Q_N(y) = \left(-g\frac{\partial}{\partial z}\right)^N e^{\frac{1}{g}(V_p(z+1)-yz)}\bigg|_{z=0}\,. \tag{2.11}$$

Now it only remains to extract its double-scaling limit.

Before proceeding to this, it is instructive to re-derive the result (2.11) using auxiliary fermions, as was done for the $(2,1)$ model in [6]. Let us introduce fermionic variables χ_i and $\chi_i^\dagger$ where i runs from 1 to N, and write

$$\det(y-B) = \int d\chi\, d\chi^\dagger e^{\chi^\dagger(y-B)\chi}\,. \tag{2.12}$$

Then, we can integrate out the matrices A and B as follows:

$$\begin{aligned} \langle \det(y-B) \rangle &= \mathcal{Z}_{(p,1)}(g)^{-1} \int dA\, dB\, d\chi\, d\chi^\dagger e^{-\frac{1}{g}\operatorname{Tr}(V_p(A+1)-AB)+\chi^\dagger(y-B)\chi} \\ &= \mathcal{Z}_{(p,1)}(g)^{-1}(2\pi g)^{N^2} \int dA\, d\chi\, d\chi^\dagger e^{-\frac{1}{g}\operatorname{Tr} V_p(A+1)+y\chi^\dagger\chi} \delta(A+g\chi\chi^\dagger) \\ &= \int d\chi\, d\chi^\dagger e^{\frac{1}{g}V_p(-g\chi^\dagger\chi+1)+y\chi^\dagger\chi}\,. \end{aligned} \tag{2.13}$$

In the last line we have used the fact that $\mathrm{Tr}(\chi\chi^\dagger)^k = -(\chi^\dagger\chi)^k$ and the fact that the partition function evaluates to

$$\mathcal{Z}_{p,1}(g) = (2\pi g)^{N^2} . \tag{2.14}$$

The next step is to introduce auxiliary parameters s and z and write

$$\begin{aligned}\langle\det(y-B)\rangle &= \int \frac{ds\,dz}{2\pi g}\int d\chi\,d\chi^\dagger\, e^{\frac{1}{g}(V_p(z+1)-yz)} e^{\frac{is}{g}(z+g\chi^\dagger\chi)} \\ &= \int \frac{ds\,dz}{2\pi g}\, e^{\frac{1}{g}(V_p(z+1)-yz+isz)}(is)^N .\end{aligned} \tag{2.15}$$

By integrating z by parts, the factor of $e^{isz/g}(is)^N$ can be converted to $e^{isz/g}(-g\partial_z)^N$ acting on the rest of the integrand. The integral over s will then give rise to a delta function constraint $\delta(z)$, and integrating over z reproduces the answer (2.11) for the expectation value of the determinant.

The derivation in terms of the fermions leads to the following physical interpretation. The matrices A and B describe the open strings between N condensed ZZ branes. These provide the open-string dual to the $(p,1)$ closed-string background, as shown (for $c=1$) in [18, 19]. Introducing the FZZT brane leads to χ and $\chi^\dagger$, which represent the open strings (including orientation) stretched between the FZZT brane and the condensed ZZ branes. Finally s and z represent the degrees of freedom for the effective theory on the FZZT brane, obtained after integrating out all the background degrees of freedom. It is interesting that s and z seem to be conjugate to one another (as do A and B). In the next subsection, we will take the double-scaling limit of the s, z integral (2.15), and we will show that it reduces to the generalized Airy function.

2.2 Double-scaling limit

In order to take the double-scaling limit, it is convenient to write (2.11) in yet another integral form

$$\langle\det(y-B)\rangle = (-g)^N N!\frac{1}{2\pi i}\oint \frac{dz}{z} e^{\frac{1}{g}(V_p(z+1)-yz)-N\log z} , \tag{2.16}$$

where the z contour integral picks up the residue of the pole at $z=0$. The integrand contains p saddle points in the complex z plane, located at the solutions of

$$(z+1)^p + yz + gN - 1 = 0 . \tag{2.17}$$

The saddles collide when $y\to 0$ and $gN=1$, and this is the critical behavior that gives rise to the double-scaling limit of the $(p,1)$ model [12]. To extract the behavior of (2.16) in the double-scaling limit, we set

$$g = \frac{1}{N} , \tag{2.18}$$

and scale

$$N = \epsilon^{-(p+1)} , \qquad z = -1+\epsilon\tilde{z} , \qquad y = \epsilon^p\tilde{y} , \tag{2.19}$$

while sending ϵ to zero. We also deform the small contour around $z = 0$ to a contour $\mathcal{C}$ along rays emanating from $z = -1$:

$$\mathcal{C}: \quad \tilde{z} = \begin{cases} -e^{-i\pi/2(p+1)}t\,, & t < 0 \\ e^{i\pi/2(p+1)}t\,, & t > 0\,. \end{cases} \tag{2.20}$$

To see that this contour deformation is valid, notice that the integrand of (2.16) goes to zero exponentially fast for $-\frac{\pi}{2(p+1)} < \arg \tilde{z} < \frac{\pi}{2(p+1)}$ and $|\tilde{z}| \to \infty$. Thus, we can close the contour (2.20) at infinity without changing the value of the integral. This closed contour encircles $z = 0$, and since the integrand of (2.16) has no other poles in the finite z plane, the contour simply picks up the residue at $z = 0$. This shows that (2.20) is indeed equivalent to the original contour around $z = 0$.

In the small ϵ limit, (2.16) becomes

$$\langle \det(y - B) \rangle = \mathcal{N}(y) \int_{\mathcal{C}} \frac{d\tilde{z}}{2\pi i}\, e^{\frac{\tilde{z}^{p+1}}{p+1} - \tilde{y}\tilde{z}} = \mathcal{N}(y) Ai_p(\tilde{y})\,, \tag{2.21}$$

where we have defined

$$Ai_p(\tilde{y}) = \int_{\mathcal{C}} \frac{d\tilde{z}}{2\pi i}\, e^{\frac{\tilde{z}^{p+1}}{p+1} - \tilde{y}\tilde{z}}\,, \tag{2.22}$$

and $\mathcal{N}(y) = \sqrt{2\pi N} N^{-1/(p+1)} e^{-N(1-y)}$ is a non-universal overall factor. Removing this factor leads to the final answer for the exact partition function $\psi(\tilde{y})$ of the FZZT brane in the $(p, 1)$ background:

$$\psi(\tilde{y}) = \lim_{N \to \infty} \mathcal{N}(y)^{-1} \langle \det(y - B) \rangle = Ai_p(\tilde{y})\,. \tag{2.23}$$

This is the generalization of the result of [6] to higher p.

In order to avoid cluttering the equations, we will henceforth drop the tildes from the double-scaled variables y and z,

$$\tilde{y} \to y\,, \qquad \tilde{z} \to z\,. \tag{2.24}$$

Since we will focus almost entirely on continuum quantities from this point on, we hope this notational change will not confuse the reader.

One recognizes that $Ai_p(y)$ satisfies the differential equation

$$\left(-\frac{\partial}{\partial y}\right)^p Ai_p(y) - y Ai_p(y) = 0\,. \tag{2.25}$$

Different choices of contours lead to different linear combinations of the homogeneous solutions of (2.25), but the particular contour (2.20) is special for the following reason. It gives rise to the unique solution of (2.25) which is real for real y and decays without oscillating at large positive y. (For $p = 2$, this function is precisely the Airy function $Ai(y)$.) It is highly nontrivial that for all p, this physical solution of (2.25) emerges unambiguously from the double-scaling limit of the two-matrix model. This is one of the main results of our paper.

By taking y to be large and positive, one recovers the asymptotic behavior

$$\log \psi(y) \approx -\frac{p}{p+1} y^{(p+1)/p} - \frac{p-1}{2p} \log y \tag{2.26}$$

which we recognize as the large y asymptotics of the FZZT disk and annulus amplitudes [4, 5]. Below, we will see that this semiclassical approximation is valid not only for large, positive y, but for *all* large y except $y \to -\infty$.

Let us also mention that, after taking the double-scaling limit, the contour of integration defining $\psi(y)$ can be deformed arbitrarily, so long as the contour does not pass through "mountains" at infinity. By "mountains", we mean asymptotic regions where the integrand grows without bound or decays slower than $|z|^{-1}$. In practice, this means that the deformed contour must asymptote to rays $-e^{-i\theta}t$ and $e^{i\theta}t$ as $t \to -\infty$ and $t \to +\infty$, respectively, with θ satisfying

$$\frac{\pi}{2(p+1)} < \theta < \frac{3\pi}{2(p+1)} \,. \tag{2.27}$$

The integral will converge fastest along the path of steepest descent $\mathcal{C}'$, which corresponds to $\theta = \frac{i\pi}{p+1}$, i.e.

$$\mathcal{C}' : \quad z \to \begin{cases} -e^{-\frac{i\pi}{p+1}} t \,, & t \to -\infty \\ e^{\frac{i\pi}{p+1}} t \,, & t \to +\infty \,. \end{cases} \tag{2.28}$$

Note that the steepest descent contour is at the same time the stationary phase contour. In the next subsection, we will use this contour to study the role of Stokes' phenomenon in the generalized Airy integral (2.22).

2.3 Asymptotic expansion and Stokes phenomenon

Using the explicit integral representation (2.22) and the saddle-point approximation, it is straightforward to analyze the large $|y|$ asymptotics of the generalized Airy functions. Just as in the case of $p = 2$, we expect to find a rich structure of Stokes lines and anti-Stokes lines. (A succinct account of Stokes' phenomenon can be found in [20].)

The integral (2.22) has p saddle points at

$$z = z_n(y) = |y|^{\frac{1}{p}} e^{\frac{i}{p}(\theta + 2\pi n)} \,, \qquad n = 0, \dots, p-1 \,, \tag{2.29}$$

where $y = |y|e^{i\theta}$. The value of the argument in the exponential at n-th saddle point is simply

$$S_n \equiv \frac{1}{p+1} z_n(y)^{p+1} - y\, z_n(y) = -\frac{p}{p+1} |y|^{\frac{p+1}{p}} e^{i\frac{p+1}{p}(\theta + 2\pi n)} \,. \tag{2.30}$$

From the form of the contour (2.20), it is not difficult to see that for large positive y only the $n = 0$ saddle contributes to the integral. The other saddles are inaccessible due to large "mountains" in the integrand of (2.22) at infinity.

As one varies y in the complex plane, however, other saddles can begin to contribute. This happens when the steepest descent contour $\mathcal{C}'$ collides with the other saddle points. The set of points in the complex y plane where this happens form what are called the

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Stokes lines. A necessary condition for this to happen is for two saddle points z_n and z_m to satisfy

$$\mathrm{Im}\, S_n = \mathrm{Im}\, S_m \tag{2.31}$$

since this means that the two saddles can be connected by a stationary phase contour. (2.31) is not a sufficient condition for Stokes phenomenon to occur, since we are only interested the particular stationary phase contour $\mathcal{C}'$ described in (2.28). In general, though, we expect from (2.31) that the first Stokes line closest to the positive, real y axis occurs at

$$\theta = \pm\frac{(p+2)\pi}{2(p+1)}\,. \tag{2.32}$$

In addition to new saddle points contributing and disappearing, one should allow for the possibility of a saddle other than $n = 0$ dominating. This would happen only when

$$\mathrm{Re}\, S_n = \mathrm{Re}\, S_0 \tag{2.33}$$

for some n. Using (2.30), this becomes

$$\cos\left(\frac{p+1}{p}(\theta + 2\pi n)\right) = \cos\left(\frac{p+1}{p}\theta\right), \tag{2.34}$$

whose only solution is $\theta = \pi$. The locus of points in the complex y plane where the dominant saddle gets exchanged with another saddle are known as the anti-Stokes lines. For the generalized Airy functions, the anti-Stokes lines are found to always lie along $\theta = \pi$. In other words, the asymptotic form of the generalized Airy function is described everywhere by the analytic continuation of the $n = 0$ saddle point, except along the negative y axis. Physically, this means that the semiclassical approximation, described in (2.26), is valid for all large y except $y \to -\infty$.

It is useful to illustrate this behavior through a concrete example. Consider the contours of fixed phase for $p = 5$. For small θ, only the dominant $n = 0$ saddle lies along the contour of steepest descent. As one increases θ from zero to $\theta = 7\pi/12$, this contour collides with the $n = 4$ contour. So for $\theta > 7\pi/12$, both saddles $n = 0$ and $n = 4$ contribute. Continuing along in this way, one finds multiple saddles contributing and disappearing, summarized in table 1.

$\theta = \arg(y)$	Contributing Saddle
$0 < \theta < \frac{7}{12}\pi$	0
$\frac{7}{12}\pi < \theta < \frac{9}{12}\pi$	0, 4
$\frac{9}{12}\pi < \theta < \frac{11}{12}\pi$	0, 3, 4
$\frac{11}{12}\pi < \theta < \pi$	0, 2, 4

Table 1: Saddle points contributing to the stationary phase integral representation of the generalized Airy function $Ai_5(y)$ as the phase of the argument y is varied.

A similar result can be obtained for $-\pi < \theta < 0$. Finally, at $\theta = \pi$, the $n = 4$ saddle replaces the $n = 0$ saddle as the dominant saddle, while at $\theta = -\pi$, the $n = 1$ saddle takes over as the dominant saddle. The structure of the saddle points and their interplay with the steepest descent contour is illustrated in figure 1. The same analysis can be repeated for general p. (The $p = 3$ case in particular was studied in [21] in a rather different context.)

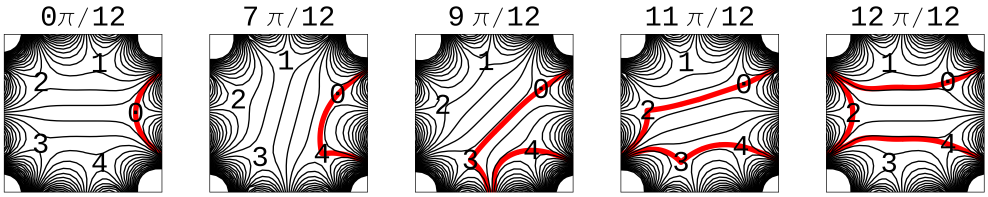

Figure 1: Contours of steepest descent in evaluating the integral expression (2.22) for the $p = 5$ generalized Airy function for various $|y| \gg 1$ are illustrated in thick (red) line. "0,1,2,3,4" label the saddle points. The phases of y are chosen to coincide with the Stoke's lines.

What is interesting about the structure of Stokes and anti-Stokes lines for generalized Airy functions is that while the identity of the dominant saddle is different between $\theta = \pi$ and $\theta = -\pi$, the actual function is smooth along the anti-Stokes line. The exact answer has no branch cut along the negative real axis of the y plane, despite the fact that the leading perturbative amplitude (2.26) is a p-sheeted cover of the y plane. Just as in the case of the $p = 2$ model, the exact FZZT amplitude is an entire function of y, and the unphysical sheets of the FZZT moduli space are eliminated by non-perturbative effects. However, as we see from table 1, the unphysical sheets still have a physical effect: they contribute small, non-perturbative corrections to the exact answer.

There is one sense in which the $p > 2$ case is different from the $p = 2$ case. The contribution from the dominant saddle (2.30) arbitrarily close to the negative real axis

$$S_0 \to -\frac{p}{p+1}|y|e^{\pm i\frac{p+1}{p}\pi} \tag{2.35}$$

has positive real part for $p > 2$. So $Ai_p(y)$ is oscillatory, but it grows exponentially in magnitude as $y \to -\infty$. Contrast this with the $p = 2$ case, where it oscillates with an amplitude which decays as a power of y. This difference between $p > 2$ and $p = 2$ has the following matrix model interpretation. As was discussed in [6] (see also section 9.3 of [22]), the oscillatory power-law decay of $Ai(y)$ as $y \to -\infty$ is directly related to the fact that the $p = 2$ model can be described by the algebraic eigenvalue density of a one-matrix model. Applying the same reasoning to $p > 2$ shows that these models *cannot* be described by an algebraic density of a single eigenvalue — attempting to construct such an eigenvalue density from $\psi(y) = Ai_p(y)$ would lead to a density that grows exponentially as $y \to -\infty$. Of course, this is the expected result since, as we saw above, these models are dual to two-matrix models which cannot be reduced to one-matrix models.

2.4 Multiple FZZT

Finally, let us show how to extend the preceding analysis of the single determinant expectation value to the case of multiple determinants. Consider a stack of m FZZT branes, whose boundary cosmological constants are described by an $m \times m$ matrix Y with eigenvalues $y_1, \ldots, y_m$. The multiple FZZT partition function in the matrix model is formulated as a double-scaling limit of

$$\Big\langle \det(Y \otimes I_N - I_m \otimes B) \Big\rangle = \frac{\det_{ij} Q_{N+i-1}(y_j)}{\Delta(y)}\,, \tag{2.36}$$

where i and j take on values $1, \ldots, m$ and $\Delta(y)$ is the Vandermonde determinant. The derivation of (2.36) can be found in e.g. [17].

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To evaluate this expression in the large-N limit, let us note that according to (2.18), a shift in N is equivalent to a shift in g. Now, g is the coupling of the lowest dimension operator, and for the $(p,1)$ models, a shift in this coupling is equivalent to a shift in y. Therefore, (2.36) becomes in the double-scaling limit

$$\Big\langle \det(Y \otimes I_N - I_m \otimes B) \Big\rangle \to \psi_m(Y) = \frac{\det_{ij} \partial_{y_i}^{j-1} \psi(y_i)}{\Delta(y)} \,. \tag{2.37}$$

Here the finite N and continuum Y are related exactly as in (2.19), and we have dropped irrelevant factors just as in (2.23). This formula for the multi-FZZT correlator is a special case of the general formula (applicable for all (p,q) models) in [6].

Using (2.23) along with the integral representation (2.22), we can turn (2.37) into

$$\psi_m(Y) = Ai_p(Y) \,, \tag{2.38}$$

where $Ai_p(Y)$ is the generalized matrix Airy function

$$Ai_p(Y) = \Delta(y)^{-1} \int_{\mathcal{C}} dz_i \, \Delta(z) e^{\frac{z_i^{p+1}}{p+1} - y_i z_i} = \int dZ \, e^{\mathrm{Tr}\left(\frac{Z^{p+1}}{p+1} - YZ\right)} \,. \tag{2.39}$$

To establish the last equality, we used an identity from [23, 24]. Eq. (2.38) shows explicitly how the effective theory on m FZZT branes in the $(p,1)$ background is described by the $m \times m$ generalized Kontsevich integral.

3. Open/closed string correspondence

The main result of the previous section can be summarized as the statement regarding the expectation value of the exponentiated macroscopic loop operator

$$\Big\langle e^{W(y)} \Big\rangle = \psi(y) = Ai_p(y) \tag{3.1}$$

in the double-scaling limit. These macroscopic loop operators, when parameterized by the length ℓ of the boundary on the world sheet, can be decomposed into local *closed-string* operators in terms of their scaling [13 − 15],

$$W(\ell) \sim \sum_{k \geq 1} \frac{\ell^{k/p}}{\Gamma\left(\frac{k+p}{p}\right)} a_{p,k} \sigma_k \,, \tag{3.2}$$

where $a_{p,k}$ is a dimensionless normalization whose precise value will be fixed in appendix B. The ℓ and the y parametrization of the macroscopic loop operators are related by Laplace transform

$$W(y) = \int \frac{d\ell}{\ell} e^{-\ell y} W(\ell) \sim \sum_{k \geq 1} \frac{p}{k} y^{-k/p} a_{p,k} \sigma_k \,. \tag{3.3}$$

A similar relation between boundaries of fixed length and local operators (as well as a Laplace transformed version) also appears in section 3.1 of [10]. One should keep in mind, however, that the expansion of macroscopic loop amplitudes in terms of microscopic operator correlation functions is subtle, in that there are divergent contributions as $\ell = 0$ or

$y \to \infty$ in the disk and the annulus amplitudes [14]. These can be removed by introducing the normalization factor

$$C(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{p}{p+1} y^{(p+1)/p} - \frac{(p-1)}{2p} \log y} , \tag{3.4}$$

which precisely cancels the divergences from the disk and the annulus amplitudes. One can then interpret the result of the previous section as the statement

$$\left\langle \exp\left(\sum_k \frac{p}{k} y^{-k/p} a_{p,k} \sigma_k \right) \right\rangle = C(y)^{-1} \psi(y) = C(y)^{-1} Ai_p(y) , \tag{3.5}$$

which relates the normalized FZZT brane partition function to the generating function of closed-string correlators in $(p, 1)$ topological string theory.

The relation (3.5) is the simplest statement of open/closed string duality — it says that the partition function of the FZZT brane is equivalent to a certain shift in the closed-string background. This relation becomes much richer if one considers multiple FZZT branes, as in section 2.4. The left hand side of (3.5) simply generalizes to

$$\left\langle \exp\left(\sum_k \frac{p}{k} \operatorname{Tr}\left(Y^{-k/p} \right) a_{p,k} \sigma_k \right) \right\rangle , \tag{3.6}$$

while the right hand side of (3.5) becomes

$$\Xi_m(Y) = C(Y)^{-1} Ai_p(Y) . \tag{3.7}$$

Here $Ai_p(Y)$ is the matrix Airy function defined in (2.39), and

$$C(Y) = \lim_{Y \to \infty} Ai_p(Y) = \int dZ \, e^{\operatorname{Tr}\left(-\frac{p}{1+p} Y^{(p+1)/p} + \frac{1}{2} \sum_{i=0}^{p-1} Y^{i/p} Z Y^{(p-i-1)/p} Z \right)} \tag{3.8}$$

with the contour of Z integration defined as in (2.39). Combining all these ingredients, we arrive at the statement of open/closed string duality

$$\left\langle \exp\left(\sum_k t_k \sigma_k \right) \right\rangle = \Xi_m(Y) , \qquad t_k = \frac{p}{k} \operatorname{Tr} Y^{-k/p} a_{p,k} , \tag{3.9}$$

which relates the generating function of topological closed string amplitudes to the generalized Kontsevich integral.

Let us note that another way of writing (3.7), which may be more familiar to some, comes from substituting $Y = X^p$

$$\Xi_m(X) = \frac{\int dZ \, e^{\operatorname{Tr}\left(\frac{Z^{p+1} - X^{p+1}}{p+1} - (Z-X) X^p \right)}}{\int dZ \, e^{\operatorname{Tr} \sum_{i=0}^{p-1} \frac{1}{2} X^i Z X^{p-i-1} Z}} . \tag{3.10}$$

This form of the generalized Kontsevich integral is common in the literature (see e.g. [16]), modulo trivial rescalings and shifts of Z and X. In the worldsheet (Liouville) description of these models, Y corresponds to the boundary cosmological constants of the FZZT branes, and X corresponds to the dual boundary cosmological constants.

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4. Discussion

In this article, we computed the exact partition function of (multiple) FZZT branes in the $(p,1)$ topological background, using the two-matrix model in the double-scaling limit. We found that these partition functions are given by the generalized Kontsevich integral. By relating this to a specific insertion of closed string operators, we were able to formulate a precise open/closed string duality for $(p,1)$ topological gravity with matter.

In principle, the double-scaled matrix model can provide a framework for understanding open/closed string duality in the most general minimal string theory. However, one should keep in mind that our analysis was aided by various simplifications that occur in the $(p,1)$ models. The most important simplification came in the calculation of the FZZT partition function at finite N. In general, the FZZT partition function is the scaling limit of an orthogonal polynomial of the dual matrix model. For the $(p,1)$ models, these orthogonal polynomials were sufficiently simple as to admit an elementary, closed-form representation. Furthermore, this representation allowed the scaling limit to be taken explicitly, which led directly to the generalized Airy function.

It would be interesting to push the matrix model analysis of open/closed string duality to more general (p,q) models which are non-perturbatively well defined. Of course, the orthogonal polynomials for the general (p,q) models are more complicated. Nonetheless, they are known to satisfy a recursion relation and can be generated in a finite number of steps. So there is still hope that one may be able to formulate concretely open/closed string duality for general (p,q).

Another open problem of immediate interest is to rederive the results in this paper using open-string field theory, as was done for $(2,1)$ in [9]. (See also [25] for recent discussion on this issue.) This would presumably shed more light both on open/closed string duality in these models, and on the inner workings of open-string field theory itself. It would also confirm the identification of the generalized Kontsevich integral with the effective theory of open strings between FZZT branes.

The primary motivation for studying explicit realizations of open/closed string duality in toy systems like $(p,1)$ is to provide new insights that can be extended to richer dynamical systems. Let us briefly mention one possible lesson that could be learned from our work. So far, all attempts to formulate purely closed-string observables from open-string field theory in general [26, 27] (see also the interesting recent work of [28]), following the work of [29 − 31], have had difficulty in removing the boundary of the world sheet. By contrast, here we encountered no such difficulties in formulating the open/closed string duality of the $(p,1)$ model: purely closed-string amplitudes were easily obtained from the generalized Kontsevich matrix integral, and the latter presumably represents a reduction of open-string field theory along the lines of [9]. Nonetheless, there was one seemingly unnatural step where we removed the divergent contribution of the disk and the annulus amplitudes by hand. Similar truncations of a few terms on the open string side also arose in the context of Gopakumar-Vafa correspondence [32]. It would be interesting if such truncations are significant and have ramifications for open/closed string dualities in general.

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A. Generalizing to other potentials

In this article we studied the two matrix model with a single multi-critical potential. However, it is very easy to generalize this discussion to superpositions of multi-critical potentials. This will allow us to use the two-matrix model to study the integrable flows between multi-critical points with different values of p, an aspect of minimal string theory that could not be studied in [6] where the one-matrix model (which always describes $p=2$) was used.

So let us consider the matrix integral

$$\mathcal{Z}(g;s_i) = \int dA\,dB\, e^{-\frac{1}{g}(V(A)+W(B)-AB)}\,, \tag{A.1}$$

where the potential is determined using the formula (2.2) of [12]

$$\begin{aligned} X_*(z) &= \frac{(z-1)^q}{z} \\ Y_*(z) &= \frac{(1-z)^p}{z} + \sum_{k=0}^{p-2} \epsilon^{-(p-k)} s_k \frac{(1-z)^k}{z}\,. \end{aligned} \tag{A.2}$$

Then the expectation value of the determinant becomes

$$\langle \det(y-B)\rangle \sim \frac{1}{2\pi i}\oint \frac{dz}{z} e^{\frac{1}{g}\left(V_p(z+1)+\sum_{k=0}^{p-2}(s_k\epsilon^{p-k}V_k(z+1))-yz\right)-N\log z}\,. \tag{A.3}$$

One can then take the double-scaling limit

$$N=\epsilon^{-p-1}\,, \qquad y=\epsilon^p\tilde{y}\,, \qquad z=-1+\epsilon\tilde{z}\,, \qquad gN = 1+\sum_{k=0}^{p-2}\epsilon^{p-k}s_k\,. \tag{A.4}$$

Then as $\epsilon\to 0$, (A.3) becomes

$$e^{-Ny}\langle\det(y-B)\rangle \sim \frac{1}{2\pi i}\int_{\mathcal{C}} d\tilde{z}\, e^{\frac{\tilde{z}^{p+1}}{p+1}+\sum_{k=1}^{p-2}s_k\frac{\tilde{z}^{k+1}}{k+1}-\tilde{y}\tilde{z}} \equiv \psi(\tilde{y})\,. \tag{A.5}$$

where we have dropped numerical factors which do not depend on y, and the contour are defined exactly as before.

One can show explicitly that the FZZT partition function (A.5) is the Baker-Akhiezer function of the KP hierarchy with Lax pair

$$Q = d^p + \sum_{k=0}^{p-1} s_k d^k\,, \qquad P = Q_+^{1/p} = d\,, \tag{A.6}$$

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where $d = \partial_{s_0} = -\partial_{\tilde{y}}$. These clearly satisfy

$$[P, Q] = 1 \tag{A.7}$$

and one can easily check that ψ satisfies the equations defining the Baker-Akhiezer function:

$$Q\psi = \tilde{y}\psi\,, \qquad P\psi = -\partial_{\tilde{y}}\psi\,. \tag{A.8}$$

Note that we can convert this to a differential equation for $\psi(\tilde{y})$ as a function of $\tilde{y}$,

$$\psi^{(p)}(\tilde{y}) + \sum_{k=0}^{p-1}(-1)^k s_k \psi^{(k)}(\tilde{y}) = (-1)^p \tilde{y}\,\psi(\tilde{y}) \tag{A.9}$$

which is a further generalization of the Airy equation.

B. Generalized Kontsevich integral and intersection numbers

Here we will fix the normalization constants $a_{p,k}$ that arose in the decomposition (3.2) of the macroscopic loop operator into closed-string scaling operators. Our method will be to match $\Xi_m(X)$ given in (3.10) with the generating function

$$\tau(t) = \exp(F) = \exp\left(\sum_{\{n,m\}} \langle \prod_{\{n,m\}} \sigma_{n,m}^{d_{n,m}} \rangle \prod_{\{n,m\}} \frac{t_{n,m}^{d_{n,m}}}{d_{n,m}!}\right) \tag{B.1}$$

of intersection numbers associated to $(p, 1)$ topological string theory. These were defined in [33] as the integral

$$\left\langle \prod_{\{m_i,n_i\}} \sigma_{n_i,m_i} \right\rangle = \frac{1}{p^g} \int_{\overline{\mathcal{M}}_{g,s}(p,m)} \prod_{i=1}^{s} c_1(\mathcal{L}_i)^{n_i} c_D(\mathcal{V}) \tag{B.2}$$

over the stable compactification of a covering $\overline{\mathcal{M}}_{g,s}(p, m)$ of the moduli space $\overline{\mathcal{M}}_{g,s}$ of a Riemann surface Σ of genus g with s distinct marked points x_i, $i = 1, \ldots, s$. Here $\sigma_{0,m}$, $m = 0, \ldots, p-2$ are the primary matter fields and $n = 0, \ldots$ labels the n-th gravitational descendant $\sigma_{n,m}$. $\mathcal{V}$ is a vector bundle over $\mathcal{M}_{g,s}(p, m)$ with fiber $H^0(C, \mathcal{R})$ of dimension $D = (g-1)(1-\frac{2}{p}) + \sum_i \frac{m_i}{p}$ and $c_D(\mathcal{V})$ is its top Chern class. The line bundle $\mathcal{R}$ is defined up to isomorphism by $\mathcal{R}^{\otimes p} = K^{p-1} \bigotimes_{i=1}^{s} \mathcal{O}(x_i)^{\otimes m_i}$, where K is the canonical bundle of Σ and $\mathcal{O}(x_i)$ are line bundles whose sections have at most simple poles at x_i. The covering $\overline{\mathcal{M}}_{g,s}(p, m)$ depends on data m, p as indicated by the notation. $c_1(\mathcal{L}_i)$ are the usual Mumford-Morita-Miller classes. On dimensional grounds the integral above is zero unless

$$\sum_{i=1}^{s}(n_i + \frac{m_i}{p} - 1) = \left(3 - \left(1 - \frac{2}{p}\right)\right)(g-1)\,. \tag{B.3}$$

A mathematically precise construction of $c_D(\mathcal{V})$ was formulated in [34, 35]. The authors of [34] further confirmed the explicit calculation of intersection numbers in [33]. A suitable compactification of $\mathcal{M}_{g,s}(p, m)$ has also been discussed in [36, 37].

As was conjectured by [10, 11] and proven for the case of $p = 2$ in [10], (B.1) is the tau function of the p reduced KP hierarchy fulfilling in addition the string equation. The geometric intersection numbers (B.2) come with natural normalization, and the first few were worked out in [33]. We shall use (B.1) with this normalization of the intersection numbers as the definition of the closed string couplings $t_{n,m}$.

Following [38], one can show that $F = \lim_{m\to\infty} \log(\Xi_m)$ fulfills the string equation and the p reduced KP hierarchy in the Miura variables $t_k \sim \mathrm{Tr}(X^{-k})$. These properties determine F up to an additive constant. One can choose the normalization of t_k so that the string equation

$$\frac{\partial F}{\partial t_{0,0}} = \frac{1}{2}\sum_{i+j=p-2} t_{0,i}t_{0,j} + \sum_{n=0}^{\infty}\sum_{m=0}^{p-2} t_{n+1,m}\frac{\partial F}{\partial t_{n,m}} \tag{B.4}$$

and the equations of the p reduced KP hierarchy

$$\frac{\partial^2 F}{\partial t_{0,0}\partial t_{n,k}} = -c_{n,k}\mathrm{res}\left(Q^{n+\frac{k+1}{p}}\right), \qquad Q = D^p - \sum_{i=0}^{p-2} u_i(\{t_{n,m}\})D^i\,, \qquad D = \frac{i}{\sqrt{p}}\frac{\partial}{\partial t_{0,0}} \tag{B.5}$$

take on precisely the form used in [33], i.e. with $c_{n,k} = (-1)^n p^{n+1}/\prod_{j=0}^{n}(jp+k+1)$. This can be achieved by setting[1]

$$t_{np+k+1} = t_{n,k} = (-p)^{\frac{k-p-n(p+2)}{2(p+1)}}\prod_{j=0}^{n-1}(jp+k+1)\,\mathrm{Tr}(X^{-(np+k+1)})\,. \tag{B.6}$$

This fixes the normalization constant $a_{p,k}$ in (3.9).

As a check of our analysis, we have calculated a few of these intersection numbers explicitly from the asymptotic expansion of $\Xi_m(X)$, and compared them with results in the literature. To extract the intersection numbers from $\Xi_m(X)$, we follow [38], the only difference being that the differential operator D with the defining properties

$$D\Xi_1(x) = \frac{\int dz\, z e^{\frac{z^{p+1}}{p+1} - zx^p + \frac{px^{p+1}}{p+1}}}{\int dz\, e^{\frac{p}{2}x^{p-1}z^2}}\,, \tag{B.7}$$

or $(D^p - x^p)\Xi_1(x) = 0$, is given in our parametrization by

$$D = \frac{p-1}{2p}\frac{1}{x^p} + x - \frac{1}{px^{p-1}}\frac{d}{dx}\,. \tag{B.8}$$

The expression for $\Xi_m(x)$ in terms of generalized characters of the symmetric group in [38] gives the exact result for all coefficients involving $\mathrm{Tr}(x^{-k})$ up to $k(p+1) \le m$. Expanding in the corresponding order and using (B.6) we obtain

$$\begin{aligned} F_{p=3} = &\left(\frac{1}{2}t_{0,0}^2 t_{0,1} + \frac{1}{12}t_{1,0}\right) + \\ &+\left(\frac{1}{72}t_{0,1}^4 + \frac{1}{2}t_{0,0}^2 t_{0,1}t_{1,0} + \frac{1}{24}t_{1,0}^2 + \frac{1}{6}t_{0,0}^3 t_{1,1} + \frac{1}{12}t_{0,0}t_{2,0}\right) + \cdots \end{aligned} \tag{B.9}$$

[1]In the case of $p = 3$, the same change of variables was described in [39].

$p=3$	$d=12$
$\langle\sigma_{0,1}^4\sigma_{1,0}\rangle_0$	$2/3$
$\langle\sigma_{0,0}^2\sigma_{0,1}\sigma_{1,0}^2\rangle_0$	2
$\langle\sigma_{0,0}^4\sigma_{2,1}\rangle_0$	1
$\langle\sigma_{0,0}\sigma_{0,1}^3\sigma_{1,1}\rangle_0$	$1/3$
$\langle\sigma_{0,0}^3\sigma_{0,1}\sigma_{2,0}\rangle_0$	1
$\langle\sigma_{0,0}^3\sigma_{1,0}\sigma_{1,1}\rangle_0$	2
$\langle\sigma_{1,0}^3\rangle_1$	$1/6$
$\langle\sigma_{0,1}\sigma_{1,1}^2\rangle_1$	$1/36$
$\langle\sigma_{0,0}\sigma_{1,0}\sigma_{2,0}\rangle_1$	$1/6$
$\langle\sigma_{0,1}^2\sigma_{2,1}\rangle_1$	$1/36$
$\langle\sigma_{0,0}^2\sigma_{3,0}\rangle_1$	$1/12$

$p=4$	$d=15$
$\langle\sigma_{0,0}^4\sigma_{2,2}\rangle_0$	1
$\langle\sigma_{0,0}^3\sigma_{0,1}\sigma_{2,1}\rangle_0$	1
$\langle\sigma_{0,0}^3\sigma_{0,2}\sigma_{2,0}\rangle_0$	1
$\langle\sigma_{0,0}^2\,\sigma_{0,1}^2\sigma_{2,0}\rangle_0$	1
$\langle\sigma_{0,0}^3\sigma_{1,0}\sigma_{1,2}\rangle_0$	2
$\langle\sigma_{0,0}\sigma_{0,1}^2\,\sigma_{0,2}\sigma_{1,2}\rangle_0$	$1/4$

$p=4$ cont.	$d=15$
$\langle\sigma_{0,1}^4\sigma_{1,2}\rangle_0$	$1/4$
$\langle\sigma_{0,0}^3\sigma_{1,1}^2\rangle_0$	2
$\langle\sigma_{0,0}^2\sigma_{0,1}\sigma_{1,0}\sigma_{1,1}\rangle_0$	2
$\langle\sigma_{0,0}\sigma_{0,1}\sigma_{0,2}^2\sigma_{1,1}\rangle_0$	$1/4$
$\langle\sigma_{0,1}^3\sigma_{0,2}\sigma_{1,1}\rangle_0$	$1/2$
$\langle\sigma_{0,0}^2\sigma_{0,2}\sigma_{1,0}^2\rangle_0$	2
$\langle\sigma_{0,0}\sigma_{0,1}^2\sigma_{1,0}^2\rangle_0$	2
$\langle\sigma_{0,1}^2\sigma_{0,2}^2\sigma_{1,0}\rangle_0$	$1/2$
$\langle\sigma_{0,2}^5\rangle_0$	$1/8$
$\langle\sigma_{0,0}\sigma_{1,2}^2\rangle_1$	$1/48$
$\langle\sigma_{0,0}^2\sigma_{3,0}\rangle_1$	$1/8$
$\langle\sigma_{0,0}\sigma_{0,2}\sigma_{2,2}\rangle_1$	$1/96$
$\langle\sigma_{0,1}^2\sigma_{2,2},\rangle_1$	$1/32$
$\langle\sigma_{0,1}\sigma_{0,2}\sigma_{2,1}\rangle_1$	$1/24$
$\langle\sigma_{0,0}\sigma_{1,0}\sigma_{2,0},\rangle_1$	$1/4$
$\langle\sigma_{0,2}^2\sigma_{2,0}\rangle_1$	$1/48$
$\langle\sigma_{0,1}\sigma_{1,1}\sigma_{1,2}\rangle_1$	$1/24$
$\langle\sigma_{0,2}\sigma_{1,0}\sigma_{1,2}\rangle_1$	$1/48$
$\langle\sigma_{1,0}^3\rangle_1$	$1/4$
$\langle\sigma_{0,2}\sigma_{1,1}^2\rangle_1$	$1/24$
$\langle\sigma_{3,2}\rangle_2$	$3/2560$

$p=5$	$d=6,12$
$\langle\sigma_{0,0}^2\sigma_{0,3}\rangle_0$	1
$\langle\sigma_{0,0}\sigma_{0,1}\sigma_{0,2}\rangle_0$	1
$\langle\sigma_{0,1}^3\rangle_0$	1
$\langle\sigma_{0,0}^3\sigma_{1,3}\rangle_0$	1
$\langle\sigma_{0,0}^2\sigma_{0,1}\sigma_{1,2}\rangle_0$	1
$\langle\sigma_{0,0}^2\sigma_{0,2}\sigma_{1,1}\rangle_0$	1
$\langle\sigma_{0,0}\sigma_{0,1}^2\sigma_{1,1}\rangle_0$	1
$\langle\sigma_{0,0}^2\,\sigma_{0,3}\sigma_{1,0}\rangle_0$	1
$\langle\sigma_{0,0}\sigma_{0,1}\sigma_{0,2}\sigma_{1,0}\rangle_0$	1
$\langle\sigma_{0,1}^3\sigma_{1,0}\rangle_0$	1
$\langle\sigma_{0,1}^2\,\sigma_{0,3}^2\rangle_0$	$1/5$
$\langle\sigma_{0,1}\sigma_{0,2}^2\sigma_{0,3}\rangle_0$	$1/5$
$\langle\sigma_{0,2}^4\rangle_0$	$2/5$
$\langle\sigma_{0,0}\sigma_{2,0}\rangle_1$	$1/6$
$\langle\sigma_{0,2}\,\sigma_{1,3},\rangle_1$	$1/60$
$\langle\sigma_{0,3}\sigma_{1,2}\rangle_1$	$1/60$
$\langle\sigma_{1,0}\rangle_1$	$1/6$
$\langle\sigma_{1,0}^2\rangle_1$	$1/6$

Table 2: Matrix model predictions for the intersection numbers defined in (B.2). The subscript indicates the genus.

and

$$\begin{aligned} F_{p=4} = {} & \left(\frac{1}{2}t_{0,0}t_{0,1}^2 + \frac{1}{2}t_{0,0}^2 t_{0,2} + \frac{1}{8}t_{1,0}\right) + \\ & + \left(\frac{1}{16}t_{0,1}^2 t_{0,2}^2 + \frac{1}{2}t_{0,0}t_{0,1}^2 t_{1,0} + \frac{1}{2}t_{0,0}^2 t_{0,2}t_{1,0} + \right. \\ & \left. + \frac{1}{16}t_{1,0}^2 + \frac{1}{2}t_{0,0}^2 t_{0,1}t_{1,1} + \frac{1}{6}t_{0,0}^3 t_{1,2} + \frac{1}{96}t_{0,2}t_{1,2} + \frac{1}{8}t_{0,0}t_{2,0}\right) + \cdots \end{aligned} \tag{B.10}$$

for $p = 3$ and $p = 4$ respectively.[2] In particular, we find agreement with the explicit calculations of [33] for the chiral ring correlators $\sigma_k = \sigma_{0,k}$

$$\langle\sigma_{k_1}\sigma_{k_2}\sigma_{k_3}\rangle = \delta_{k_1+k_2+k_3,p-2}\langle\sigma_{k_1}\sigma_{k_2}\sigma_{k_3}\sigma_{k_4}\rangle = \frac{1}{p}\min(k_i, p-1-k_i)\,, \tag{B.11}$$

from which all chiral ring correlators on the sphere follow by associativity. We also agree with $\langle\tau_{1,0}\rangle = (p-1)/24$. More generally we find that for $p \leq 10$ and $k \leq 3$, all predicted intersection numbers are positive rational numbers. Further predictions for the intersection numbers with homogeneous degree d where $\sigma_{n,k}$ has degree $pn + k + 1$ are tabulated in table 2 for $p = 3, 4, 5$.

[2]For $p = 2$ our expansion agrees with the one in [38].

Finally, let us emphasize that, despite all the progress in making Witten's conjecture mathematically precise [34 − 37], a proof of (B.1) is still lacking for $p > 2$. That is, it remains to extend beyond $p = 2$ Kontsevich's statement that the intersection numbers for $(p, 1)$ topological gravity are generated by the tau function of the p reduced KP hierarchy. Such a generalization will presumably involve open string field theory techniques in order to generate the cell decomposition of the moduli space of spin curves. See [9, 25] for a recent discussion on this issue.

References

[1] N. Seiberg and D. Shih, *Minimal string theory*, *Comptes Rendus Physique* **6** (2005) 165 [hep-th/0409306].

[2] V. Fateev, A.B. Zamolodchikov and A.B. Zamolodchikov, *Boundary Liouville field theory, I. Boundary state and boundary two-point function*, hep-th/0001012.

[3] J. Teschner, *Remarks on Liouville theory with boundary*, hep-th/0009138.

[4] N. Seiberg and D. Shih, *Branes, rings and matrix models in minimal (super)string theory*, *JHEP* **02** (2004) 021 [hep-th/0312170].

[5] D. Kutasov, K. Okuyama, J. Park, N. Seiberg and D. Shih, *Annulus amplitudes and ZZ branes in minimal string theory*, *JHEP* **08** (2004) 026 [hep-th/0406030].

[6] J. Maldacena, G.W. Moore, N. Seiberg and D. Shih, *Exact vs. semiclassical target space of the minimal string*, *JHEP* **10** (2004) 020 [hep-th/0408039].

[7] M. Aganagic, R. Dijkgraaf, A. Klemm, M. Mariño and C. Vafa, *Topological strings and integrable hierarchies*, hep-th/0312085.

[8] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, *Surveys Diff. Geom.* **1** (1991) 243.

[9] D. Gaiotto and L. Rastelli, *A paradigm of open/closed duality: Liouville D-branes and the Kontsevich model*, hep-th/0312196.

[10] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix airy function*, *Commun. Math. Phys.* **147** (1992) 1.

[11] E. Witten, *The N matrix model and gauged WZW models*, *Nucl. Phys.* **B 371** (1992) 191.

[12] J.M. Daul, V.A. Kazakov and I.K. Kostov, *Rational theories of 2-d gravity from the two matrix model*, *Nucl. Phys.* **B 409** (1993) 311 [hep-th/9303093].

[13] T. Banks, M.R. Douglas, N. Seiberg and S.H. Shenker, *Microscopic and macroscopic loops in nonperturbative two-dimensional gravity*, *Phys. Lett.* **B 238** (1990) 279.

[14] G.W. Moore, N. Seiberg and M. Staudacher, *From loops to states in 2-d quantum gravity*, *Nucl. Phys.* **B 362** (1991) 665.

[15] G.W. Moore and N. Seiberg, *From loops to fields in 2-d quantum gravity*, *Int. J. Mod. Phys.* **A 7** (1992) 2601.

[16] P. Di Francesco, P.H. Ginsparg and J. Zinn-Justin, *2-d gravity and random matrices*, *Phys. Rept.* **254** (1995) 1 [hep-th/9306153].

[17] A. Morozov, *Integrability and matrix models*, *Phys. Usp.* **37** (1994) 1 [hep-th/9303139].

[18] J. McGreevy and H. Verlinde, *Strings from tachyons: the $c = 1$ matrix reloaded*, *JHEP* **12** (2003) 054 [hep-th/0304224].

[19] I.R. Klebanov, J. Maldacena and N. Seiberg, *D-brane decay in two-dimensional string theory*, *JHEP* **07** (2003) 045 [hep-th/0305159].

[20] M.V. Berry, *Stokes' phenomenon smoothing a Victorian discontinuity*, *Inst. Hautes Études Sci. Publ. Math.* **68** (1988) 211.

[21] L. Fidkowski, V. Hubeny, M. Kleban and S. Shenker, *The black hole singularity in AdS/CFT*, *JHEP* **02** (2004) 014 [hep-th/0306170].

[22] P.H. Ginsparg and G.W. Moore, *Lectures on 2-d gravity and 2-d string theory*, hep-th/9304011.

[23] Harish-Chandra, *Differential operators on a semisimple Lie algebra*, *Am. J. Math.* **79** (1957) 87.

[24] C. Itzykson and J.B. Zuber, *The planar approximation, 2*, *J. Math. Phys.* **21** (1980) 411.

[25] S. Giusto and C. Imbimbo, *The Kontsevich connection on the moduli space of FZZT Liouville branes*, *Nucl. Phys.* **B 704** (2005) 181 [hep-th/0408216].

[26] A. Hashimoto and N. Itzhaki, *Observables of string field theory*, *JHEP* **01** (2002) 028 [hep-th/0111092].

[27] D. Gaiotto, L. Rastelli, A. Sen and B. Zwiebach, *Ghost structure and closed strings in vacuum string field theory*, *Adv. Theor. Math. Phys.* **6** (2003) 403 [hep-th/0111129].

[28] M. Baumgartl, I. Sachs and S.L. Shatashvili, *Factorization conjecture and the open/closed string correspondence*, hep-th/0412266.

[29] J.A. Shapiro and C.B. Thorn, *BRST invariant transitions between closed and open strings*, *Phys. Rev.* **D 36** (1987) 432.

[30] J.A. Shapiro and C.B. Thorn, *Closed string-open string transitions and Witten's string field theory*, *Phys. Lett.* **B 194** (1987) 43.

[31] B. Zwiebach, *Interpolating string field theories*, *Mod. Phys. Lett.* **A 7** (1992) 1079 [hep-th/9202015].

[32] R. Gopakumar and C. Vafa, *On the gauge theory/geometry correspondence*, *Adv. Theor. Math. Phys.* **3** (1999) 1415 [hep-th/9811131].

[33] E. Witten, *Algebraic geometry associated with matrix models of two dimensional gravity*, in proceedings of *Topological Methods in Modern Mathematics*, Stony Brook, NY 1991, Publish or Perish, Houston 1993.

[34] T.J. Jarvis, T. Kimura and A. Vaintrob, *Moduli spaces of higher spin curves and integrable hierarchies*, *Compos. Math.* **126** (2001) 157 [math.AG/9905034].

[35] A. Polishchuk, *Witten's top Chern class on the moduli space of higher spin curves*, math.AG/0208112.

[36] D. Abramovich and T. Jarvis, *Moduli of twisted spin curves*, *Proc. Amer. Math. Soc.* **131** (2003) 685 [math.AG/0104154].

[37] H. Fan, T. Jarvis and Y. Ruan, *Geometry and analysis of spin equations*, math.DG/0409434.

[38] C. Itzykson and J.B. Zuber, *Combinatorics of the modular group, 2. The Kontsevich integrals*, *Int. J. Mod. Phys.* **A 7** (1992) 5661 [hep-th/9201001].

[39] C. Kristjansen, *Multiloop correlators for rational theories of 2-d gravity from the generalized Kontsevich models*, *Nucl. Phys.* **B 436** (1995) 342 [hep-th/9409066].